10.1 Introduction

1) In Chap. 8, we derived the NSE and developed several exact solutions.
2) In this Chapter, we will study several methods for simplifying the NSE, which permit use of mathematical analysis and solution: these approximations often hold for certain regions of the flow field.
10.2 Nondimensional N-S Equation

1) **Purpose:** Order-of-magnitude analysis of the terms in the N-S equation, which is necessary for simplification and approximate solutions.

2) **Incompressible NSE:**

\[
\rho \frac{D \vec{V}}{Dt} = \rho \left( \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right) = -\nabla P + \mu \nabla^2 \vec{V} + \rho \vec{g}
\]

3) Each term is *dimensional*, and each variable or property (\(\rho, \ V, \ t, \ \mu, \) etc.) is also *dimensional*.

4) **What are the primary dimensions of each term in the N-S equation?**

5) To nondimensionalize, choose **scaling parameters** as follows

<table>
<thead>
<tr>
<th>TABLE 10-1</th>
</tr>
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<tbody>
<tr>
<td><strong>Scaling Parameter</strong></td>
</tr>
<tr>
<td>(L)</td>
</tr>
<tr>
<td>(V)</td>
</tr>
<tr>
<td>(t)</td>
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<tr>
<td>(P_0 - P_\infty)</td>
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<tr>
<td>(g)</td>
</tr>
</tbody>
</table>

6) Every additive term has primary dimensions \(\{m^1L^{-2}T^{-2}\}\). To nondimensionalize, multiply every term by \(L/(\rho V^2)\), which has primary dimensions \(\{m^{-1}L^2T^2\}\), so that the dimensions cancel. After rearrangement:

\[
\left[ \frac{fL}{V} \right] \frac{\partial \vec{V}^*}{\partial t^*} + \left( \vec{V}^* \cdot \nabla^* \right) \vec{V}^* = - \left[ \frac{P_0 - P_\infty}{\rho V^2} \right] \nabla^* P^* + \left[ \frac{gL}{V^2} \right] \vec{g}^* + \left[ \frac{\mu}{\rho V L} \right] \nabla^* \vec{V}^*
\]
10.2 Nondimensional N-S Equation

Terms in [ ] are nondimensional parameters:

\[ \frac{fL}{V} \frac{\partial \tilde{V}^*}{\partial t^*} + (\tilde{V}^* \cdot \nabla^*) \tilde{V}^* = - \left[ \frac{P_0 - P_\infty}{\rho V^2} \right] \nabla^* P^* + \left[ \frac{gL}{V^2} \right] \tilde{g}^* + \left[ \frac{\mu}{\rho V L} \right] \nabla^{*2} \tilde{V}^* \]

Navier-Stokes equation in Nondimensional Form:

\[ [St] \frac{\partial \tilde{V}^*}{\partial t^*} + (\tilde{V}^* \cdot \nabla^*) \tilde{V}^* = - [Eu] \nabla^* P^* + \left[ \frac{1}{Fr^2} \right] \tilde{g}^* + \left[ \frac{1}{Re} \right] \nabla^{*2} \tilde{V}^* \]

10.3 Creeping Flow

1) Also known as “Stokes Flow” or “Low Reynolds number flow”
2) Occurs when Re << 1
   ① \rho, V, or L are very small, e.g., micro-organisms, MEMS, nano-tech, particles, bubbles
   ② \mu is very large, e.g., honey, lava
3) To simplify NSE, assume St \sim 1, Fr \sim 1

\[ [Eu] \nabla^* P^* = \left[ \frac{1}{Re} \right] \nabla^{*2} \tilde{V}^* \]

Pressure forces \hspace{1cm} Viscous forces

4) Since \ P^* \sim 1, \ \nabla^* \sim 1

\[ Eu = \frac{P_0 - P_\infty}{\rho V^2} \sim \frac{1}{Re} = \frac{\mu}{\rho V L} \quad \Rightarrow \quad P_0 - P_\infty \sim \frac{\mu V}{L} \]
10.3 Creeping Flow

1) This is important

\[ P_0 - P_\infty \sim \frac{\mu V}{L} \]

① Very different from inertia dominated flows where

\[ P_0 - P_\infty \sim \rho V^2 \]

② Density has completely dropped out of NSE. To demonstrate this, convert back to dimensional form.

\[ \nabla P = \mu \nabla^2 \bar{V} \]

③ This is now a **LINEAR EQUATION** which can be solved for simple geometries.

2) Solution of Stokes flow is beyond the scope of this course.

3) **Analytical solution for flow over a sphere** gives a drag coefficient which is a linear function of velocity V and viscosity \( \mu \).

\[ F_D = 3\pi \mu V D \]

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**10.3 Creeping Flow**

Flowing 1mm/s between glass plates spaced 1mm apart.  
Re = 2,000

Creeping Flow (Hele-Shaw Flow)  
Laminar & Turbulent Flow

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10.3 Creeping Flow

10.4 Inviscid Regions of Flow

1) Inviscid Regions: where net viscous forces are negligible compared to pressure and/or inertia forces

2) Euler Equation:

\[ \text{~0 if } Re \text{ large} \]

3) Euler equation often used in aerodynamics and hydrodynamics

4) Elimination of viscous term changes PDE from mixed elliptic-hyperbolic to hyperbolic. This affects the type of analytical and computational tools used to solve the equations.

5) Must “relax” wall boundary condition from no-slip to slip

- No-slip BC: \( u = v = w = 0 \)
- Slip BC: \( \tau_w = 0, V_n = 0 \)

\( V_n \) = normal velocity
10.4.1 Potential Flow

1) Irrotational approximation: vorticity is negligibly small
\[ \zeta = \nabla \times \vec{V} = 2\tilde{\omega} \approx 0 \]

2) In general, inviscid regions are also irrotational.

3) Continuity equation
   ① Use the vector identity: \( \nabla \times \nabla \phi = 0 \)
   ② Since the flow is irrotational: \( \nabla \times \vec{V} = 0 \)
   ③ \( \phi \) is a scalar (velocity) potential function
\[ \vec{V} = \nabla \phi \]

4) Regions of irrotational flow are also called regions of potential flow.

5) Ideal Flow = Incompressible + Inviscid Flow

6) Potential Flow = Ideal Flow + Irrotational Flow

10.4.2 Potential Flow: velocity potential

1) Coordinate Systems
   ① Cartesian coordinates:
\[ \vec{V}(u,v,w) = \nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \]
   ② Polar coordinates
\[ \vec{V}(u_r,u_\theta,u_z) = \frac{\partial \phi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{e}_\theta + \frac{\partial \phi}{\partial z} \hat{e}_z \]

2) Substituting into the continuity equation gives: Laplace Equation
\[ \nabla \cdot \vec{V} = 0 \quad \rightarrow \quad \nabla \cdot \nabla \phi = \nabla^2 \phi = 0 \]

3) This means we only need to solve 1 linear scalar equation to determine all 3 components of velocity!

4) Laplace equation appears in numerous fields of science, engineering, and mathematics. This means there are well developed tools for solving this equation.
10.4.2 Potential Flow: stream function

1) In general, continuity equation cannot be used by itself to solve for flow field, however it can be used to find the missing velocity component if the flow field is incompressible.

2) Consider the continuity equation for an incompressible 2D flow:

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

3) Definition of the Stream Function: one dependent variable ($\psi$) instead of two dependent variables ($u, v$)

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

4) Substitution 3) into 2) yields: which is identically satisfied for any smooth function $\psi(x, y)$.

$$\frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left( -\frac{\partial \psi}{\partial x} \right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$
10.4.2 Potential Flow: stream function

1) **Difference in** $\psi$ **from one streamline to another is equal to the volume flow rate per unit width between the two streamlines**

$$\vec{n} = \frac{dy}{ds} \vec{i} - \frac{dx}{ds} \vec{j}$$

$$d\dot{\psi} = \vec{V} \cdot \vec{n} \, dA = (u \vec{i} + v \vec{j}) \cdot \left( \frac{dy}{ds} \vec{i} - \frac{dx}{ds} \vec{j} \right) \, ds$$

$$d\dot{\psi} = u \, dy - v \, dx = \frac{\partial \psi}{\partial y} \, dy + \frac{\partial \psi}{\partial x} \, dx = d\psi$$

$$\dot{\psi}_B = \int_B \vec{V} \cdot \vec{n} \, dA = \int_B d\dot{\psi} = \int_{\psi=\psi_2}^{\psi=\psi_1} d\psi = \psi_2 - \psi_1$$

10.4.2 Potential Flow: Momentum Equation

1) **Momentum equation**

   ① If we can compute $\phi$ from the Laplace equation (which came from continuity) and velocity from the definition $\vec{V} = \nabla \phi$, why do we need the NSE? Answer: To compute Pressure.

   ② Apply irrotational approximation to viscous term of the NSE:

   $$\mu \nabla^2 \vec{V} = \mu \nabla^2 (\nabla \phi) = \mu \nabla (\nabla^2 \phi) = 0$$

2) Therefore, the NSE reduces to the Euler equation for irrotational flow:

$$\rho \left( \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right) = -\nabla P + \mu \nabla^2 \vec{V} + \rho \vec{g}$$

3) Instead of integrating to find $P$, use vector identity to derive Bernoulli equation:

$$(\vec{V} \cdot \nabla) \vec{V} = \nabla \left( \frac{\vec{V} \cdot \vec{V}}{2} \right) - \vec{V} \times (\nabla \times \vec{V}) = \nabla \left( \frac{V^2}{2} \right) - \vec{V} \times \vec{\omega}$$
10.4.2 Potential Flow: Momentum Equation

1) **Steady Euler equation** to be written as

\[ \nabla \left( \frac{V^2}{2} \right) \! - \! \vec{V} \! \times \! \vec{\zeta} = -\frac{\nabla P}{\rho} + \vec{g} \]

2) This form of Bernoulli equation is valid for inviscid and irrotational flow since we’ve shown that NSE reduces to the Euler equation.

\[ \nabla \left( \frac{P}{\rho} + \frac{V^2}{2} + gz \right) = \vec{V} \! \times \! \vec{\zeta} \]

3) **Inviscid Flow: along a streamline**

4) **Irrotational Flow: everywhere**

\[ \frac{P}{\rho} + \frac{V^2}{2} + gz = C \]

10.4.2 Potential Flow: Stream Function & Velocity Potential

1) Velocity components and Laplace Eqn. in **Cartesian coordinates**:

\[ u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \]

\[ \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \]

2) Velocity components and Laplace Eqn. in **Cylindrical coordinates**:

\[ u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} \]

\[ \nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \theta^2} = 0 \]
10.4.2 Potential Flow: Stream Function & Velocity Potential

1) Recall the definition of streamfunction:
   ① Constant values of $\psi$: streamlines
   ② Constant values of $\phi$: equipotential lines
   ③ $\psi$ and $\phi$ are mutually orthogonal
   ④ $\psi$ and $\phi$ are harmonic functions
   ⑤ $\psi$ is defined by continuity; $\nabla^2\psi$ results from irrotationality
   ⑥ $\phi$ is defined by irrotationality; $\nabla^2\phi$ results from continuity

2) Since vorticity is zero:
   \[ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \]

3) Laplace equation holds for the streamfunction and the velocity potential:

   \[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \rightarrow \frac{\partial^2 \psi}{\partial x^2} \pm \frac{\partial^2 \psi}{\partial y^2} = 0 \]

Flow solution can be achieved by solving either $\nabla^2\phi$ or $\nabla^2\psi$, however, BC are easier to formulate for $\psi$. 

10.4.2 Potential Flow: Stream Function & Velocity Potential

1) Streamlines: $d\psi=0$ ($\psi$ is constant along streamlines)

   \[ d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \]

   \[ \frac{dy}{dx} \bigg|_{\psi=0} = \frac{-\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} = \frac{v}{u} \]

2) Equipotential lines: $d\phi=0$

   \[ d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0 \]

   \[ \frac{dy}{dx} \bigg|_{\phi=0} = \frac{-\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} = \frac{-u}{v} \]

Thus $\psi$ and $\phi$ are mutually orthogonal 

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10.5 2D Potential Flows

1) Method of Superposition
   ① Since $\nabla^2 \phi = 0$ is linear, a linear combination of two or more solutions is also a solution, e.g., if $\phi_1$ and $\phi_2$ are solutions, then $(A\phi_1) + (A+\phi_1)$, $(\phi_1 + \phi_2)$, $(A\phi_1 + B\phi_2)$ are also solutions.
   ② Also true for $y$ in 2D flows ($\nabla^2 \psi = 0$).
   ③ Velocity components are also additive:

\[
u = \frac{\partial \phi}{\partial x} = \frac{\partial (\phi_1 + \phi_2)}{\partial x} = \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial x}\]

2) Given the principal of superposition, there are several elementary planar irrotational flows which can be combined to create more complex flows.
   ① Uniform stream
   ② Line source/sink
   ③ Line vortex
   ④ Doublet

10.5.1 Uniform Stream

1) In Cartesian coordinates

\[V(u,v) = u\hat{i} + 0\hat{j}\]

\[u = \frac{\partial \phi}{\partial x} = V, \quad v = \frac{\partial \phi}{\partial y} = 0\]

\[\phi = \int Vdx + c = Vx\]

\[u = \frac{\partial \psi}{\partial y} = V, \quad v = -\frac{\partial \psi}{\partial x} = 0\]

\[\psi = \int Vdy + c = V\psi\]

2) In Cylindrical coordinates

\[x = r\cos \phi \rightarrow \phi = Vr\cos \theta\]

\[y = r\sin x \rightarrow \psi = Vr\sin \theta\]
10.5.2 Line Source

1) Potential and streamfunction are derived by observing that **volume flow rate across any circle**: 

2) Velocity components: 

\[ \phi = \frac{Q}{2\pi} \ln r \quad u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{Q}{2\pi} \frac{1}{r} \]

\[ \psi = \frac{Q}{2\pi} \theta \quad u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = 0 \]

3) If the source is placed at \((a, b)\)

\[ \phi = \frac{Q}{2\pi} \ln r_i = \frac{Q}{2\pi} \ln \sqrt{(x-a)^2 + (y-b)^2} \]

\[ \psi = \frac{Q}{2\pi} \theta = \frac{Q}{2\pi} \tan^{-1} \left( \frac{(y-b)}{(x-a)} \right) \]

10.5.2 Line Source & Sink

1) Stream function:

\[ \frac{\partial \psi}{\partial r} = -u_\theta = 0 \quad \therefore \psi = f(\theta) \]

\[ \frac{\partial \psi}{\partial \theta} = f'(\theta) = ru_r = \frac{Q}{2\pi} \]

\[ \therefore \psi = \frac{Q}{2\pi} \theta + c \quad (c = 0) \]

2) Source strength: volume flow rate: \( \frac{\dot{V}}{L} \quad [m^2/s] \)

\[ \int u_r dl = \int_0^{2\pi} u_r r d\theta = \int_0^{2\pi} \frac{Q}{2\pi} d\theta = Q \]

3) Sink: \( Q < 0 \)

\[ \phi = -\frac{Q}{2\pi} \ln r, \quad \psi = \frac{Q}{2\pi} \theta \]
10.5.3 Line Vortex

1) Vortex at the origin. Velocity components

\[ u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = 0 \quad u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r} \]

2) These can be integrated to give \( \phi \) and \( \psi \)

\[ \phi = \frac{\Gamma}{2\pi} \theta, \quad \psi = -\frac{\Gamma}{2\pi} \ln r \theta \]

3) If vortex is moved to \((x, y) = (a, b)\)

\[ \phi = \frac{\Gamma}{2\pi} \theta = \frac{\Gamma}{2\pi} \tan^{-1}\left(\frac{y-b}{x-a}\right) \]

\[ \psi = -\frac{\Gamma}{2\pi} r_i = -\frac{\Gamma}{2\pi} \ln \sqrt{(x-a)^2 + (y-b)^2} \]

10.5.4. Superposition (Uniform + Source)

1) Superposition of uniform and source flows:

\[ \phi = Vr \cos \theta + \frac{Q}{2\pi} \ln r \]

2) Velocity

\[ u_r = \frac{\partial \phi}{\partial r} = V \cos \theta + \frac{Q}{2\pi} \frac{1}{r}, \quad u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -V \sin \theta \]

3) Stagnation point \((r_{stag})\)

\[ u_r = u_\theta = 0 \quad (at \ \theta = \pi) \]

\[ u_r(\pi)_{stag} = -V + \frac{Q}{2\pi} r_{stag} = 0, \quad u_\theta(\pi) = 0 \]

\[ \therefore r_{stag} = \frac{Q}{2\pi V} \]

4) Stream function

\[ \psi = V \cos \theta + \frac{Q}{2\pi} \pi + \frac{Q}{2} \]

\[ \psi_{stag} = 0 + \frac{Q}{2}\pi = \frac{Q}{2} \]

\[ V + \frac{Q}{2\pi} \theta = \frac{Q}{2} \pi \rightarrow y = \frac{Q}{2\pi V} (\pi - \theta) = \frac{Q}{2V} \left(1 - \frac{\theta}{\pi}\right) \]
10.5.4. Superposition (Source + Sink = Doublet)

1) Velocity potential

\[ \phi = \frac{\Lambda}{2\pi} \lim_{a \to 0} \left( \ln \left( \frac{x^2 + y^2}{(x-a)^2 + y^2} \right) \right) \]

\[ = \frac{\Lambda}{2\pi} \lim_{a \to 0} \frac{x}{\left( \frac{1}{2} \frac{x^2 + y^2}{(x-a)^2 + y^2} \right)} \]

\[ = \frac{\Lambda}{2\pi} \frac{x}{\left( \frac{1}{2} \frac{r}{r} \right)} \]

\[ \therefore \phi = \frac{\Lambda}{2\pi} \frac{\cos \theta}{r} \]

2) Stream function

\[ u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \]

\[ u_r = -\frac{\Lambda}{2\pi} \frac{\cos \theta}{r^2} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \]

\[ \frac{\partial \psi}{\partial \theta} = -\frac{\Lambda}{2\pi} \frac{\cos \theta}{r} \]

\[ \therefore \psi = -\frac{\Lambda}{2\pi} \frac{\sin \theta}{r} \]
10.5.5 Superposition (Sink + Vortex)

1) Superposition of sink and vortex: Spiral vortex

\[
\psi = \frac{Q}{2\pi} \theta - \frac{\Gamma}{2\pi} \ln r
\]

\[
u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{Q}{2\pi} \frac{1}{r}
\]

\[
u_\theta = -\frac{1}{r} \frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}
\]

10.5.6 Superposition (Uniform + Doublet)

1) Flow over a circular cylinder: Free stream + doublet

\[
\phi = Vr \cos \theta + \frac{\Lambda \cos \theta}{2\pi r}, \quad \psi = Vr \sin \theta - \frac{\Lambda \sin \theta}{2\pi r}
\]

2) Assume body is \(\psi = 0\) (at \(r = a\))

\[
\psi_{body} = \sin \theta \left(V - \frac{\Lambda}{2\pi r}\right) = 0 \quad \Rightarrow \quad \Lambda = 2\pi a^2 V
\]

\[
\phi = Vr \cos \theta + \frac{a^2 V \cos \theta}{r} = V \cos \theta \left(r + \frac{a^2}{r}\right)
\]

\[
\psi = Vr \sin \theta - \frac{a^2 V \sin \theta}{r} = V \sin \theta \left(r - \frac{a^2}{r}\right)
\]

3) Velocity fields:

\[
u_r = \frac{\partial \phi}{\partial r} = V \cos \theta \left(1 - \frac{a^2}{r^2}\right)
\]

\[
u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -V \sin \theta \left(1 + \frac{a^2}{r^2}\right)
\]
10.5.6 Superposition (Uniform + Doublet)

1) On the cylinder surface (r=a)
\[ u_r = \frac{\partial \phi}{\partial r} = V \cos \theta \left( 1 - \frac{a^2}{r^2} \right) = 0 \]
\[ u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -V \sin \theta \left( 1 + \frac{a^2}{r^2} \right) = -2V \sin \theta \]

2) Compute pressure using Bernoulli equation and velocity on cylinder surface
\[ \frac{P}{\rho} + \frac{V^2}{2} + gV = \frac{P_\infty}{\rho} + \frac{V_\infty^2}{2} + gV_\infty \]
\[ C_p = \frac{P - P_\infty}{\rho V_\infty^2} \left( 1 - \frac{V^2}{V_\infty^2} \right) \]
\[ u_r = 0, \quad u_\theta = -2V_\infty \sin \theta \]
\[ V^2 = u_r^2 + u_\theta^2 = 4V_\infty^2 \sin^2 \theta \]
\[ \therefore C_p = 1 - 4 \sin^2 \theta = 1 - 4 \sin^2 \beta \]

10.5.7 Velocity on cylinder surface

\[ u_r = 0, \quad u_\theta = -2V_\infty \sin \theta \]
\[ V^2 = u_r^2 + u_\theta^2 = 4V_\infty^2 \sin^2 \theta \]
\[ \therefore C_p = 1 - 4 \sin^2 \theta = 1 - 4 \sin^2 \beta \]
10.5.8 D’Alembert’s Paradox

1) **D’Alembert’s Paradox**: Integration of surface pressure (which is symmetric in $x$), reveals that the **drag is zero**.

\[
P = P_x + \frac{\rho V_x^2}{2} - \frac{\rho V_x^2}{2} = P_x + \frac{\rho V_x^2}{2} - 2\rho V_x^2 \sin^2 \theta
\]

\[
\begin{aligned}
F_x &= -\int P \cos \theta dA = -a \int_0^{2\pi} P \cos \theta d\theta = 0 \\
F_y &= -\int P \sin \theta dA = -a \int_0^{2\pi} P \sin \theta d\theta = 0
\end{aligned}
\]

2) For the irrotational flow approximation, the drag force on any non-lifting body of any shape immersed in a uniform stream is **zero**.

3) Why?
   ① Viscosity and the viscous effects have been neglected.
   ② No-slip condition are responsible for:
      a. Flow separation (pressure drag)
      b. Wall-shear stress (friction drag)

10.6 Boundary Layer (BL) Approximation

1) BL approximation bridges the gap between the Euler and NS equations, and between the slip and no-slip BC at the wall.

2) Prandtl (1904) introduced the BL approximation.
10.6.1 Boundary Layer (BL) Approximation

1) BL Equations: we restrict attention to steady, 2D, laminar flow (although method is fully applicable to unsteady, 3D, turbulent flow)

2) BL coordinate system
   ① \( x \): tangential direction
   ② \( y \): normal direction

10.6.2 Boundary Layer (BL) Approximation

1) Steady nondimensional NS equations
\[
\left( \frac{\partial \tilde{v}^*}{\partial x} + (\tilde{v}^* \cdot \nabla^*) \tilde{v}^* \right) \tilde{v}^* = -\left[ Eu \right] \nabla^* P^* + \left[ \frac{1}{Re} \right] \nabla^* v^* 
\]

2) Recall definitions
\[
Eu = \frac{P_0 - P_\infty}{\rho V^2}, \quad Re = \frac{\rho V L}{\mu}
\]

3) Since \( \rho V^2 \sim P - P_\infty \) : \( Eu \sim 1 \)

4) \( Re \gg 1 \), Should we neglect viscous terms? No!, because we would end up with the Euler equation along with deficiencies already discussed.

5) Can we neglect some of the viscous terms?

6) To answer question, we need to redo the nondimensionalization
   ① Use \( L \) as length scale in stream-wise direction and for derivatives of velocity and pressure with respect to \( x \).
   ② Use \( \delta \) (boundary layer thickness) for distances and derivatives in \( y \).
   ③ Use local outer (or edge) velocity \( U_e \).
\[
y = \delta \implies U = U_e(x)
\]
10.6.3 Boundary Layer (BL) Approximation

1) Orders of Magnitude:

\[ U \sim 1, \quad P - P_\infty \sim \rho U^2, \quad \frac{\partial}{\partial x} \sim \frac{1}{L}, \quad \frac{\partial}{\partial y} \sim \frac{1}{\delta} \]

2) Continuity equation:

\[ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \]

\[ \sim \frac{U}{L} \sim \frac{V}{\delta} \]

3) Since \( \delta/L \ll 1 \rightarrow V \ll U \)

4) Define new nondimensional variables

\[ x^* = \frac{x}{L}, \quad y^* = \frac{y}{\delta}, \quad U^* = \frac{U}{U_e}, \quad V^* = \frac{V L}{U_e \delta}, \quad P^* = \frac{P - P_\infty}{\rho U_e^2} \]

5) All are order unity, therefore normalized.

6) Apply to \( x \)- and \( y \)-components of NSE.

7) Go through details of derivation on blackboard.

10.6.4 Boundary Layer (BL) Approximation

1) Incompressible Laminar Boundary Layer Equations

1) Continuity equation:

\[ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \]

2) X-Momentum equation:

\[ U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = U_e \frac{\partial U_e}{\partial x} + \nu \frac{\partial^2 U}{\partial y^2} \]

3) Y-Momentum equation:

\[ \frac{\partial P}{\partial y} = 0 \]

2) Solve for outer flow:

using Potential Flow approach, ignoring the BL.

3) Assume \( \delta/L \ll 1 \) (thin BL) and solve BLE:

1) \( y = 0 \) \( \Rightarrow \) no-slip, \( u=0, \ v=0 \)

2) \( y = \delta \) \( \Rightarrow \) \( U = U_e(x) \)

3) \( x = x_0 \Rightarrow u = u(x_0), \ v=v(x_0) \)

4) Calculate \( \delta, \theta, \delta^*, \tau_w, \) Drag

5) Verify \( \delta/L \ll 1 \)

6) If \( \delta/L \) is not \( \ll 1 \), use \( \delta^* \) as body and go to step 1 and repeat
9.5.5 Navier-Stokes Equation

1) **Continuity Equation:**

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

Calculate velocity \((U, V, W)\) and pressure \((P)\) for known geometry with Boundary Conditions (BC), and Initial Conditions (IC).

2) **Momentum Equations:**

\[
x: \quad \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x
\]

\[
y: \quad \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho g_y
\]

\[
z: \quad \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \rho g_z
\]

10.7 Boundary Layer Procedure

1) **Possible Limitations**

- \(Re\) is not large enough \(\Rightarrow\) BL may be too thick for thin BL assumption.
- due to wall curvature \(\delta \sim R\), \(\partial p/\partial y \neq 0\)
- \(Re\) too large \(\Rightarrow\) turbulent flow at \(Re = 1 \times 10^5\). BL approximation still valid, but new terms required.
- Flow separation

2) Before defining and \(\delta^*\) and \(\theta\), are there analytical solutions to the BL equations?
   - Unfortunately, NO

3) **Blasius Similarity Solution** boundary layer on a flat plate, constant edge velocity, zero external pressure gradient
10.8 Blasius Similarity Solution

1) Blasius introduced similarity variables

\[ f' = \frac{U}{U_e} \quad \eta = y \sqrt{\frac{U_e}{\nu x}} \]

2) This reduces the BLE to

\[ f''' + f f'' = 0 \]

\[ f(0) = f'(0) = 0, \quad f'(\infty) = 1 \]

3) This ODE can be solved using Runge-Kutta technique.

4) Result is a BL profile which holds at every station along the flat plate.

10.8.1 Blasius Similarity Solution

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( f' )</th>
<th>( f'' )</th>
<th>( f''' )</th>
<th>( \eta )</th>
<th>( f' )</th>
<th>( f'' )</th>
<th>( f''' )</th>
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<td>0.00000</td>
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<td>0.03321</td>
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<td>0.06641</td>
<td>0.00664</td>
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<td>0.18401</td>
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<td>1.00000</td>
<td>8.27921</td>
</tr>
</tbody>
</table>

* \( \eta \) is the similarity variable defined in Eq. 4 above, and function \( f(\eta) \) is solved using the Runge-Kutta numerical technique. Note that \( f' \) is proportional to the shear stress \( \tau \), \( f'' \) is proportional to the \( x \)-component of velocity in the boundary layer (\( f' = \alpha /U \)), and \( f' \) itself is proportional to the stream function. \( f' \) is plotted as a function of \( \eta \) in Fig. 10-99.
10.8.2 Blasius Similarity Solution

1) Boundary layer thickness can be computed by assuming that $\delta$ corresponds to point where $U/U_e = 0.990$. At this point, $\eta = 4.91$, therefore

$$\eta = 4.91 = \sqrt{\frac{U_e}{\nu x}} \Rightarrow \frac{\delta}{x} = \frac{4.91}{\sqrt{Re_x}}$$

2) Wall shear stress $\tau_w$ and friction coefficient $C_{f,x}$ can be directly related to Blasius solution

$$\tau_w = \mu \frac{\partial U}{\partial y} \bigg|_{y=0} = f''(0) \frac{\rho U_e^2}{\sqrt{Re_x}} = 0.332 \frac{\rho U_e^2}{\sqrt{Re_x}}$$

$$C_{f,x} = \frac{\tau_w}{\frac{1}{2} \rho U_e^2} = \frac{0.664}{\sqrt{Re_x}}$$

10.9 Displacement Thickness

1) Displacement thickness $\delta^*$ is the imaginary increase in thickness of the wall (or body), as seen by the outer flow, and is due to the effect of a growing BL.

2) Expression for $\delta^*$ is based upon control volume analysis of conservation of mass

$$\delta^* = \int_0^\infty \left(1 - \frac{U}{U_e}\right) dy$$

3) Blasius profile for laminar BL can be integrated to give

$$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{Re_x}} \Rightarrow (\approx 1/3 \text{ of } \delta)$$
10.10 Momentum Thickness

1) Momentum thickness $\theta$ is another measure of boundary layer thickness.

2) Defined as the loss of momentum flux per unit width divided by $\rho U^2$ due to the presence of the growing BL.

3) Derived using CV analysis:

$$\theta = \int_{0}^{\infty} \frac{U}{U_e} \left(1 - \frac{U}{U_e}\right) \, dy = \frac{F_{D,x}}{\rho U_e^2 \delta x}$$

$$\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}$$

$\theta$ for Blasius solution, identical to $C_{f,x}$

10.11 Turbulent Boundary Layer

1) All BL variables [$U(y)$, $\delta$, $\delta^*$, $\theta$] are determined empirically.

2) One common empirical approximation for the time-averaged velocity profile is the one-seventh-power law

$$y \leq \delta \quad \frac{U}{U_e} = \left(\frac{y}{\delta}\right)^{1/7} \quad y > \delta \quad \frac{U}{U_e} \approx 1$$
10.11 Turbulent Boundary Layer

<table>
<thead>
<tr>
<th>Property</th>
<th>Laminar</th>
<th>(a)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary layer thickness</td>
<td>( \frac{\delta}{x} = \frac{4.91}{\sqrt{Re}} )</td>
<td>( \frac{\delta}{x} = \frac{0.16}{(Re)^{0.77}} )</td>
<td>( \frac{\delta}{x} = \frac{0.38}{(Re)^{0.5}} )</td>
</tr>
<tr>
<td>Displacement thickness</td>
<td>( \frac{\delta}{x} = \frac{1.72}{\sqrt{Re}} )</td>
<td>( \frac{\delta}{x} = \frac{0.020}{(Re)^{0.77}} )</td>
<td>( \frac{\delta}{x} = \frac{0.048}{(Re)^{0.5}} )</td>
</tr>
<tr>
<td>Momentum thickness</td>
<td>( \frac{\theta}{x} = \frac{0.664}{\sqrt{Re}} )</td>
<td>( \frac{\theta}{x} = \frac{0.016}{(Re)^{0.77}} )</td>
<td>( \frac{\theta}{x} = \frac{0.037}{(Re)^{0.5}} )</td>
</tr>
<tr>
<td>Local skin friction coefficient</td>
<td>( C_{f,s} = \frac{0.664}{\sqrt{Re}} )</td>
<td>( C_{f,s} \approx \frac{0.027}{(Re)^{0.77}} )</td>
<td>( C_{f,s} \approx \frac{0.059}{(Re)^{0.5}} )</td>
</tr>
</tbody>
</table>

*Laminar values are exact and are listed to three significant digits, but turbulent values are listed to only two significant digits due to the large uncertainty affiliated with all turbulent flow fields.

† Obtained from one-seventh-power law.

‡ Obtained from one-seventh-power law combined with empirical data for turbulent flow through smooth pipes.

1) Flat plate zero-pressure-gradient TBL can be plotted in a universal form if a new velocity scale, called the friction velocity \( U_f \), is used.

2) Law of the Wall:

\[
\begin{align*}
\upsilon^+ &= \frac{U}{U_f} y^+ &= \frac{U_f y}{\nu} \\
U_f &= \sqrt{\frac{\tau_w}{\rho}}
\end{align*}
\]

3) Despite its simplicity, the Law of the Wall is the basis for many CFD turbulence models.

4) Spalding (1961) developed a formula which is valid over most of the boundary layer (\( \kappa, B \) are constants)

\[
y^+ = u^+ e^{\kappa B} \left[ e^{\kappa u^+} - 1 - \kappa u^+ - \frac{(\kappa u^+)^2}{2} - \frac{(\kappa u^+)^3}{6} \right]
\]
10.12 Pressure Gradients

1) Shape of the BL is strongly influenced by external pressure gradient
   ① favorable (dP/dx < 0)
   ② zero
   ③ mild adverse (dP/dx > 0)
   ④ critical adverse (τ_w = 0)
   ⑤ large adverse with reverse (or separated) flow

2) Turbulent BL is more resistant to flow separation than laminar BL exposed to the same adverse pressure gradient

Laminar flow separates at corner
Turbulent flow does not separate